Self-Adjoint Vector Variational Formulation for Lossy Anisotropic Dielectric Waveguide

SRBOLJUB R. CVETKOVIC, STUDENT MEMBER, IEEE, AND J. BRIAN DAVIES, MEMBER, IEEE

***Abstract*—This paper presents the derivation of a new self-adjoint variational formula for complex propagation constant in a lossy anisotropic dielectric waveguide, in terms of the magnetic field and real frequency. The ability to include loss and anisotropy (into the permittivity tensor) while preserving the self-adjointness of the system is achieved by using the less common real-type inner product. When used as a basis of Rayleigh–Ritz or finite-element methods, the formula leads to the canonical eigenvalue equation of the form $Ax = \gamma^2 Bx$.**

I. Introduction

BECAUSE OF THE design needs for optical and microwave guides, research has increasingly applied computer methods for solving uniform waveguiding structures such as optical channel guide and microwave image guide. Finite elements have featured particularly in the analysis of guides where the permittivity varies arbitrarily in the transverse plane, the permittivity simply varying from element to element [1]–[4].

This paper is concerned with the problem where the material is lossy, and permittivity, as well as varying transversely, may be of arbitrary but preferably symmetric tensor form. Recent finite-element work dealing with the loss has used weighted-residual methods that i) are not variational [5]–[8], ii) do not result in canonical eigenvalue matrix form [5]–[8], and iii) involve solutions via a complex frequency [5]–[8].

This paper presents a new variational formula for the complex propagation constant explicitly in terms of real frequency and the 3-vector H. When used in a finite-element or Rayleigh–Ritz procedure, this formula has three respective advantages of i) giving complex $\gamma = \alpha + j\beta$ in a stationary form (with improved accuracy in γ for a given field representation), ii) resulting in a standard eigenvalue matrix form

$$\boldsymbol{A}\boldsymbol{x} = \gamma^2 \boldsymbol{B}\boldsymbol{x} \tag{1}$$

and so allowing the solution by well-established algorithms [13], and iii) giving the above matrix form directly for the required real frequency and so avoiding the unnecessary iteration via complex frequencies [5], [6], [8].

As with the earlier references [5]–[8], concern here is with loss that is not particularly small, However, another application of the variational formula is to the class of structures already described, but where the loss is small. A more efficient solution for γ would then follow by the standard perturbation technique, i.e., solving for the simpler loss-free case first [1]–[4] and then substituting into the variational formula.

The first variational formula is for permittivities that are complex but symmetric so that the resulting waveguide is lossy but reciprocal [14]. A different and more general formula (also derived in the paper) emerges when a quite arbitrary permittivity tensor is included to allow guiding that is both lossy and nonreciprocal. The problem ceases to be self-adjoint, so that the "original" and a separate "adjoint" magnetic field both need separate representation, thus doubling the matrix order. Further, in place of (1), the matrix equation becomes

$$\gamma^2 \boldsymbol{A}\boldsymbol{x} + \gamma \boldsymbol{B}\boldsymbol{x} + \boldsymbol{C}\boldsymbol{x} = 0. \tag{2}$$

This nonstandard eigenvalue problem can "routinely" be reduced to (1) via the theory of "lambda" matrices [12], but the resulting matrix order is doubled again. We, therefore, emphasize that it is computationally important to preserve the self-adjoint system if at all possible.

The procedure we have adopted for deriving the variational formula is classical, and has been most recently discussed by Chen and Lien [9]. The details of the analysis are, however, specific to our problem. Our account also includes a little about the crucial choice of real or complex inner products when applied to self-adjoint systems, as well as a discussion on physical interpretation of the adjointness relationship. Hence, the paper is divided into two main sections: in Section II, some important aspects of the derivation will be discussed, and in Section III, the derivation itself will be presented, including the outline of the proof of stationarity.

II. Self-Adjointness and the Choice of Inner Products

The object of this study is to establish a variational (or stationary) formula for the computation of fields in a passive uniform dielectric waveguide, shown in Fig. 1. Provided that such a formula can indeed be found, the independent field components will then be represented by a complete set of guided modes (eigenvectors) at a given frequency, as well as the propagation constant (eigenvalue)

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The authors are with the Department of Electronic and Electrical Engineering, University College London, London WC1E 7JE, England.
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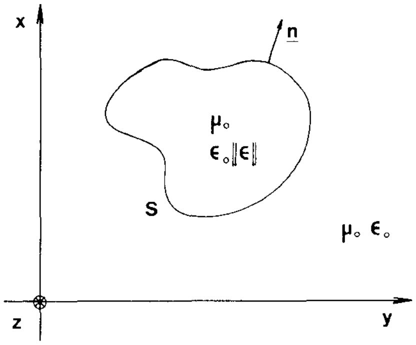

Fig. 1. The geometry of the cross section of a passive uniform dielectric waveguide.

associated with each mode. The guide dispersion characteristic can then be obtained easily by using the same formula at different frequencies.

The first step towards the solution is an introduction of an adjoint waveguide, as a problem auxiliary to the original one. This idea comes from the physical interpretation of the variational principle using the concept of generalized reactions between the original and adjoint systems [9]–[11]. Even though it provides a mathematical basis for the variational solution of electromagnetic problems, the relationship between the original and the adjoint guides is based on the orthogonality between the modes in a real, physical waveguide [10]. We will, therefore, consider the adjointness in detail.

Assuming field time-dependence $\exp(j\omega t)$ (from now on the term being implied), the following Maxwell equations are used to describe the vector fields in the original waveguide (Fig. 1):

$$-\nabla \times \mathcal{E} = j\omega\mu_0\mathcal{H} \tag{3}$$

$$\nabla \times \mathcal{H} = j\omega\epsilon_0\|\epsilon\|\mathcal{E}. \tag{4}$$

Similarly, Maxwell's equations for the fields in the adjoint waveguide can be written as

$$-\nabla \times \mathcal{E}^a = j\omega\mu_0\mathcal{H}^a \tag{5}$$

$$\nabla \times \mathcal{H}^a = j\omega\epsilon_0\|\epsilon^a\|\mathcal{E}^a \tag{6}$$

In (3)–(6), μ_0 and ϵ_0 are the parameters of free space and $\|\epsilon\|$ and $\|\epsilon^a\|$ are the relative complex permittivity tensors for the media filling the original and the adjoint guide, respectively. (Note that the vector fields $\mathcal{E}, \mathcal{E}^a$, $\mathcal{H}$, and $\mathcal{H}^a$, are also complex.) The relative permeability of dielectrics is assumed to be unity.

Even though we are essentially working in three dimensions, the structure of the uniform waveguides is effectively two dimensional. Hence, the constraint imposed on the permittivity tensors is that they must be independent of the axial coordinate (z in this case) so that the media is only transversely inhomogeneous. At this stage, we will not impose any other restrictions on the guiding media. Thus, it can also be lossy and anisotropic and the permittivity tensor will, therefore, (in the matrix form) look like

$$\|\epsilon(x,y)\| = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

where all terms may be complex.

Also due to the two-dimensional structure, the waves will have z-dependence $\exp(-\gamma z)$. We can then rewrite $\mathcal{E}$, $\mathcal{E}^a$, $\mathcal{H}$, and $\mathcal{H}^a$ in the form

$$\mathcal{E}(x,y,z) = \boldsymbol{E}(x,y)\exp(-\gamma z)$$

$$\mathcal{E}^a(x,y,z) = \boldsymbol{E}^a(x,y)\exp(-\gamma^a z) \tag{7}$$

$$\mathcal{H}(x,y,z) = \boldsymbol{H}(x,y)\exp(-\gamma z)$$

$$\mathcal{H}^a(x,y,z) = \boldsymbol{H}^a(x,y)\exp(-\gamma^a z) \tag{8}$$

where γ and γ^a are the propagation constants corresponding to the original and adjoint fields, respectively, and are complex numbers of the form

$$\gamma = \alpha + j\beta. \tag{9}$$

We begin discussing adjointness by establishing the relationship between γ and γ^a. But first, we need to define a surface real inner product of the two complex vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ as

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \iint_R \boldsymbol{A}\cdot\boldsymbol{B}\,dx\,dy \tag{10}$$

where the region of integration R is the cross section of the guide. The surface complex inner product is defined similarly, $\boldsymbol{B}$ being replaced by $\boldsymbol{B}^*$, the complex conjugate. The inner product mapping defined above has its standard properties, summarized in [9] and [15].

As it will be shown later, the choice between the real and complex inner products is arbitrary *only* for the most general class of problems (i.e., the nonself-adjoint ones). However, for the self-adjoint cases, important restrictions have to be imposed on the media tensors (and linear operators), depending on which inner product definition is chosen.

Whenever the inner product is used in this work, vector $\boldsymbol{A}$ is to be replaced by the original field or its linear transform, and vector $\boldsymbol{B}$ by the adjoint field (or its transform). Hence, to obtain useful expressions by evaluating the surface integral, the product $\boldsymbol{A}\cdot\boldsymbol{B}$ must only be a function of x and y, i.e., the exponential (z-dependent) terms of the original $(\mathcal{E},\mathcal{H})$ and the adjoint $(\mathcal{E}^a,\mathcal{H}^a)$ vector fields must cancel out. This is possible only if:

$$\gamma^a = -\gamma \text{ (for real inner products), or} \tag{11}$$

$$\gamma^a = \gamma^* \text{(for complex inner products).} \tag{12}$$

Note that conditions (11) and (12) apply to both self-adjoint and nonself-adjoint systems.

From the relationship of γ and γ^a (eq. (11) and (12)), it can be seen that for a mode in the original guide, associated with γ and propagating in the positive z-direction, the corresponding mode in the adjoint guide with $-\gamma^a$ as its eigenvalue, will propagate in the negative z direction. However, in the case of a self-adjoint system, γ (and similarly γ^a) occurs simultaneously as an eigenvalue in the original and the adjoint guide. This follows from the fact that the roots of the standard eigenvalue equation (2) come in pairs symmetric about the origin in the complex plane. On the other hand, the nonself-adjoint formula for the propagation constant results in a nonstandard eigenvalue equation (2), whose roots do not possess such symmetry.

Now we re-express Maxwell's equations in linear operator form, with (3)–(6) becoming

$$\mathscr{L}f = 0 \tag{13}$$

$$\mathscr{L}^a f^a = 0 \tag{14}$$

where f and f^a are the original and adjoint electric and/or magnetic fields, and $\mathscr{L}$ and $\mathscr{L}^a$ are their respective linear operators. To obtain $\mathscr{L}$ for the magnetic field, for example, we have to express the vector field $\mathscr{E}$ from (4) in terms of vector $\mathscr{H}$:

$$\mathscr{E} = \|\epsilon\|^{-1} \frac{\nabla \times \mathscr{H}}{j\omega\epsilon_0} \tag{15}$$

and substitute it into (3), which then becomes

$$\mathscr{L}\mathscr{H} \equiv \nabla \times \left(\|\epsilon\|^{-1} \nabla \times \mathscr{H}\right) - \omega^2 \mu_0 \epsilon_0 \mathscr{H} = 0. \tag{16}$$

We can similarly obtain $\mathscr{L}^a$ from the equations for the adjoint field:

$$\mathscr{L}^a \mathscr{H}^a \equiv \nabla \times \left(\|\epsilon^a\|^{-1} \nabla \times \mathscr{H}^a\right) - \omega^2 \mu_0 \epsilon_0 \mathscr{H}^a = 0. \tag{17}$$

$\mathscr{L}^a$ can also be found directly from $\mathscr{L}$ by simply replacing $\|\epsilon\|$ by its adjoint $\|\epsilon^a\|$ in (16). Two remarks should be made about $\|\epsilon\|^{-1}$.

i) The inverse of the adjoint permittivity tensor is equal to the adjoint of the inverse tensor, or

$$\left(\|\epsilon^a\|\right)^{-1} = \left(\|\epsilon\|^{-1}\right)^a. \tag{18}$$

ii) It can be shown that if we split $\|\epsilon\|^{-1}$ into its real and imaginary parts, the imaginary part would again, as with $\|\epsilon\|$, correspond to losses.

We could equally deduce $\mathscr{L}$ for an electric field, or indeed for a 6-vector field representing both vectors $\mathscr{E}$ and $\mathscr{H}$. In general, the identity that relates the fields in the original and the adjoint systems to their respective operators is

$$\langle \mathscr{L}^a f^a, f \rangle = \langle f^a, \mathscr{L} f \rangle. \tag{19}$$

The operators $\mathscr{L}$ and $\mathscr{L}^a$ can be subject to a number of boundary conditions. We are particularly interested in the following:

a) the electric wall or $\boldsymbol{n} \times \mathscr{E} = \boldsymbol{n} \times \mathscr{E}^a = 0$

b) the magnetic wall or $\boldsymbol{n} \times \mathscr{H} = \boldsymbol{n} \times \mathscr{H}^a = 0$

c) the radiation boundary condition (unbounded system)

where $\boldsymbol{n}$ is an outward normal unit vector to the waveguide wall S (Fig. 1). The radiation boundary condition represents the case when the cross section of the guide extends to infinity, one way of dealing with this being to use the so-called "infinite" elements [4].

Conditions a) and b) can be either essential or natural; for this study, we assume them to be essential boundary conditions (which is easy to arrange), so that the trial fields f and f^a will indeed satisfy a) or b). Then we do not need to include any additional terms into the derivation of the stationary formula, as in [9]. In the case of a), it is assumed that we have either a physical symmetry plane(s) or the boundaries of the guide are perfect conductors. If loss needs to be introduced into the waveguide wall, we can represent it by a lossy dielectric which extends to infinity, our boundary condition now being c). The importance of condition b) is purely computational, as it occurs only when applied to planes of physical symmetry.

Due to the fact that the given boundary conditions are always self-adjoint (unlike, for example, the "impedance" wall), the geometry of cross sections of the original and adjoint guides is identical.

To deduce the relationship between the transversely varying parts of the eigenmodes/vector fields, namely $\boldsymbol{E}$ and $\boldsymbol{E}^a$ (or $\boldsymbol{H}$ and $\boldsymbol{H}^a$), we observe the relationship between the characteristics of the guiding media in the original and the adjoint systems, as well as their respective boundary conditions. For the nonself-adjoint problem, $\|\epsilon\|$ and $\|\epsilon^a\|$ are related by

$$\|\epsilon^a\| = \|\epsilon\|^T \quad \text{(for real inner products)} \tag{20}$$

$$\|\epsilon^a\| = \left(\|\epsilon\|^T\right)^* \quad \text{(for complex inner products).} \tag{21}$$

From (20) and (21) and Maxwell's equations, it can be shown that the transversely varying parts of the original and the adjoint fields are not simply related at all. However, for the self-adjoint problems, the media in the original and the adjoint guide are, by definition, the same, i.e.,:

$$\|\epsilon^a\| = \|\epsilon\|. \tag{22}$$

Thus, we have the following two relations, in addition to (20) and (21) above:

$$\|\epsilon\| = \|\epsilon\|^T \quad \text{(for real inner products)} \tag{23}$$

$$\|\epsilon\| = \left(\|\epsilon\|^T\right)^* \quad \text{(for complex inner products).} \tag{24}$$

In this case, the fields in the two guides will be represented by two identical sets of eigenvectors (which can, again, be deduced from Maxwell's equations, (23) and (24) above, and also the fact that γ occurs simultaneously in both guides). Hence, for the self-adjoint case, we have

$$\boldsymbol{H}(x, y) = \boldsymbol{H}^a(x, y) \tag{25}$$

$$\boldsymbol{E}(x, y) = \boldsymbol{E}^a(x, y). \tag{26}$$

Thus, the adjoint field does not need to be represented separately, which, in turn, means that the required matrix order is halved. Therefore, the self-adjoint problems are computationally very desirable. But there is a price to pay. The medium can be lossy *providing* its tensor is symmetric (as in (23), the guide then being reciprocal [14]). Alternatively its tensor can be nonsymmetric *providing* it is Hermitian (viz, loss-free, as in (24), the guide then being nonreciprocal). To preserve the self-adjoint system, a constraint must be accepted, and a correct choice of inner product made. The case of concern in this paper involves a lossy dielectric which is reciprocal. Hence, to derive the stationary formula, we must use the real inner product.

Finally, we consider the ways in which an electromagnetic field can be represented in a variational form. We consider the choice of formulation in relation to our specific problem of a transversely inhomogeneous dielectric. Four possible formulations are

a) $E-H$ formulation (6-vector)
b) E formulation (3-vector)
c) H formulation (3-vector)
d) E_z-H_z formulation (2-vector).

The formulations a) and b) are not appealing since some transverse components of E are discontinuous at the boundary of two dielectrics. Any attempt to take care of this physical discontinuity would result in an unnecessary complication, especially knowing that all components of H (as well as the z component of E) are continuous over the cross section. The choice is narrowed down to c) and d).

From the aspect of computer storage, choice d) seems to be the more desirable, the field being represented by the minimum number of components. However, it can be shown that for a dielectric with a general symmetric tensor, the resulting formulation would give a nonstandard eigenvalue problem. As mentioned before, this can be reduced to a standard eigenvalue problem at the expense of doubling the matrix order. Therefore, with the H-formulation for a given memory space or matrix order, we have the most compact field representation.

III. Variational Formulation

To derive the stationary formula for the propagation constant, we begin by obtaining a more general variational functional I.

The first variation δI of this functional must be equal to zero. For the simultaneous solution of (13) and (14), the idea (based on the principle of generalized reactions [9] and [11]) is to express δI in the form

$$\delta I \equiv \langle \delta f^a, \mathscr{L} f \rangle + \langle \mathscr{L}^a f^a, \delta f \rangle \tag{27}$$

where δf and δf^a are the variations of f and f^a, respectively. The functional I can now be found from δI by simply "integrating" the right-hand side of (27), with respect to the unknown fields f and f^a, i.e.,

$$I = \langle f^a, \mathscr{L} f \rangle \tag{28}$$

or, using the identity (19) governing the use of linear operators, the equivalent of (28) is:

$$I = \langle \mathscr{L}^a f^a, f \rangle \tag{29}$$

and for the self-adjoint case, (29) simply reduces to

$$I = \langle \mathscr{L} f^a, f \rangle. \tag{30}$$

We derive the functional for the nonself-adjoint case first, and then deduce from it the self-adjoint formulation (30) as its special case. To obtain the formula for any stationary quantity that might be contained in the functional I, we use the additional fact that

$$I(f, f^a) = 0 \tag{31}$$

which is only true for the passive (sourceless) media. Having decided on the choice of a real inner product and the H formulation, we now substitute (17) into the above (31), which then becomes (using the fact that the curl operator is symmetric)

$$\langle \|\epsilon^a\|^{-1} \nabla \times \mathscr{H}^a, \nabla \times \mathscr{H} \rangle - \omega^2 \mu_0 \epsilon_0 \langle \mathscr{H}^a, \mathscr{H} \rangle = 0 \tag{32}$$

or if the alternative expression (29) for I is used, we have

$$\langle \nabla \times \mathscr{H}^a, \|\epsilon\|^{-1} \nabla \times \mathscr{H} \rangle - \omega^2 \mu_0 \epsilon_0 \langle \mathscr{H}^a, \mathscr{H} \rangle = 0. \tag{33}$$

Two possible parameters for which the variational formula can be deduced from I (or more precisely from (31)) are the frequency and the propagation constant. The formula for the frequency directly follows from (32)

$$\omega^2 = \frac{\langle \|\epsilon^a\|^{-1} \nabla \times \mathscr{H}^a, \nabla \times \mathscr{H} \rangle}{\mu_0 \epsilon_0 \langle \mathscr{H}^a, \mathscr{H} \rangle}.$$

Note that the above formula is nonself-adjoint, so that it applies to any dielectric tensor. Nevertheless, it still reduces to the standard eigenvalue problem, unlike the general formula for the propagation constant. However, the frequency obtained for a given propagation constant is complex. For the time-harmonic solution, we then have to iterate until the imaginary part becomes sufficiently small [5], [6], [8]. This has to be done even for the self-adjoint cases. As mentioned before, this iteration is wasteful when compared with the directness of the solution that can be achieved with a variational formula for the complex propagation constant.

To deduce a formula for the propagation constant from (32) or (33), we split the curl operator into transverse and longitudinal parts

$$\nabla \times \equiv \nabla_T \times + \nabla_z \times \tag{34}$$

where $\nabla_T \times$ and $\nabla_z \times$ are

$$\nabla_T \times = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} \qquad \nabla_z x = \frac{\partial}{\partial z}\hat{z}$$

and $\hat{x}$, $\hat{y}$, and $\hat{z}$ unit vectors in the x, y and z directions, respectively. If we substitute (34) into (32) and differentiate w.r.t. z only, using (11), we arrive at

$$\begin{aligned} &\langle \|\epsilon^a\|^{-1} \nabla_T \times \mathscr{H}^a, \nabla_T \times \mathscr{H} \rangle \\ &- \gamma \langle \|\epsilon^a\|^{-1} \nabla_T \times \mathscr{H}^a, \hat{z} \times \mathscr{H} \rangle \\ &+ \gamma \langle \|\epsilon^a\|^{-1} \hat{z} \times \mathscr{H}^a, \nabla_T \times \mathscr{H} \rangle \\ &- \gamma^2 \langle \|\epsilon^a\|^{-1} \hat{z} \times \mathscr{H}^a, \hat{z} \times \mathscr{H} \rangle \\ &- \omega^2 \mu_0 \epsilon_0 \langle \mathscr{H}^a, \mathscr{H} \rangle = 0 \end{aligned} \tag{35}$$

which is the nonself-adjoint formula for the propagation constant. Clearly, (35) is a nonstandard eigenvalue equation (2).

Taking into account the fact that the solutions of the original and the adjoint waveguide for self-adjoint cases are two identical sets of eigenvalues and eigenvectors, as well as (23), it can be shown that (35) reduces to

$$\gamma^2 = \frac{\langle \|\epsilon\|^{-1}\nabla_T \times \boldsymbol{H}, \nabla_T \times \boldsymbol{H}\rangle - \omega^2\mu_0\epsilon_0\langle \boldsymbol{H}, \boldsymbol{H}\rangle}{\langle \|\epsilon\|^{-1}\hat{z} \times \boldsymbol{H}, \hat{z} \times \boldsymbol{H}\rangle}. \quad (36)$$

Note that the fields have been replaced with their transversely varying parts (eq. (8)), as the z-dependent terms cancel in the inner product. Now, expressing the inner products in (36) in their integral form, we finally obtain

$$\gamma^2 = \frac{\iint \|\epsilon\|^{-1}(\nabla_T \times \boldsymbol{H})\cdot(\nabla_T \times \boldsymbol{H})\, dx\, dy - \omega^2\mu_0\epsilon_0 \iint \boldsymbol{H}\cdot\boldsymbol{H}\, dx\, dy}{\iint \|\epsilon\|^{-1}\hat{z} \times \boldsymbol{H}\cdot\hat{z} \times \boldsymbol{H}\, dx\, dy} \quad (37)$$

as the self-adjoint formula for the propagation constant involving lossy reciprocal media.

Now to check that the obtained formula (37) is indeed stationary, we rewrite it as

$$\langle \|\epsilon\|^{-1}\nabla_T \times \boldsymbol{H}, \nabla_T \times \boldsymbol{H}\rangle - \omega^2\mu_0\epsilon_0\langle \boldsymbol{H}, \boldsymbol{H}\rangle = \gamma^2\langle \|\epsilon\|^{-1}\hat{z} \times \boldsymbol{H}, \hat{z} \times \boldsymbol{H}\rangle. \quad (38)$$

If the small perturbation $\delta\boldsymbol{H}$ of the vector field $\boldsymbol{H}$ is introduced into (38), it results in a small change $\delta\gamma^2$ in the term γ^2 at a given frequency. Hence, (38) becomes

$$\begin{aligned}&-(\gamma^2 + \delta\gamma^2)\langle \|\epsilon\|^{-1}\hat{z} \times (\boldsymbol{H} + \delta\boldsymbol{H}), \hat{z} \times (\boldsymbol{H} + \delta\boldsymbol{H})\rangle \\ &\quad - \omega^2\mu_0\epsilon_0\langle \boldsymbol{H} + \delta\boldsymbol{H}, \boldsymbol{H} + \delta\boldsymbol{H}\rangle \\ &\quad + \langle \|\epsilon\|^{-1}\nabla_T \times (\boldsymbol{H} + \delta\boldsymbol{H}), \nabla_T \times (\boldsymbol{H} + \delta\boldsymbol{H})\rangle = 0. \quad (39)\end{aligned}$$

If the propagation constant term γ^2 is stationary, then $\delta\gamma^2$ must be zero by Euler's theorem. Hence, if we subtract (38) from (39), ignore all terms of second order, and put $\delta\gamma^2$ to zero, we arrive at

$$\langle \delta\boldsymbol{H}, \nabla \times (\|\epsilon\|^{-1}\nabla \times \boldsymbol{H}) - \omega^2\mu_0\epsilon_0\boldsymbol{H}\rangle = 0. \quad (40)$$

For an arbitrary $\delta\boldsymbol{H}$ in (40), the second term of the inner product emerges as the Euler equation (41)

$$\nabla \times (\|\epsilon\|^{-1}\nabla \times \boldsymbol{H}) - \omega^2\mu_0\epsilon_0\boldsymbol{H} = 0 \quad (41)$$

thus completing the proof of stationarity.

IV. Conclusion

A new variational formula (eq. (37)) has been established for the solution of passive, uniform dielectric waveguides, with an arbitrary permittivity profile. By using the less common real inner product, the desired variational formulation has been achieved in self-adjoint form, in terms of real frequency, and leading to standard eigenvalue matrix form. This has been achieved by accepting the constraint of a symmetric (complex) permittivity tensor. Although a new formulation, it is in effect complementary to the well-known formulas (e.g., [16]) which used a complex inner product, and so (for self-adjointness) was constrained to a Hermitian permittivity tensor.

For the rare situation concerning material tensors that are neither Hermitian nor symmetric (where the guide could be lossy and nonreciprocal), a nonself-adjoint version of the variational formula is given in (35), though here the choice of real or complex inner product is no longer crucial.

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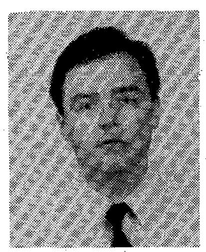

Srboljub R. Cvetkovic (S'84) was born on August 5, 1959 in Belgrade, Yugoslavia. He received the B. Sc. degree with first class honors in electronic engineering from the City University, London, England, in 1983. He is currently working towards the Ph.D. degree in electronic engineering on a computer solution of optical and microwave guides, at University College London, England.

Mr. Cvetkovic is a member of Eta Kappa Nu, and an associate member of the Institute of Electrical Engineers, London, England.

J. Brian Davies (M'73) was born in Liverpool, England, in 1932. From Jesus College, Cambridge, he received a degree in mathematics in 1955. From the University of London, England, he received the M.Sc. degree in mathematics, the Ph.D. degree in mathematical physics, and the D.Sc. (Eng.) degree in engineering in 1957, 1960, and 1980, respectively.

From 1955 to 1963, he worked at the Mullard Research Laboratories, Salford, Surrey, England, except for two years spent at University College, London, England. In 1963, he joined the Staff of the Department of Electrical Engineering, University of Sheffield, England. Since 1967, he has been on the Staff at University College, London, where he is now Professor in Electrical Engineering. From 1971 to 1972, he was a Visiting Scientist at the National Bureau of Standards, Boulder, CO. In 1983, he was a Visitor at the Department of Engineering Science, University of Oxford, England. His research interests have been with problems of electromagnetic theory, especially those requiring computer methods of solution. But recently, his interests have extended into the field problems of acoustic microscopy and of surface acoustic waves.

Prof. Davies is a Member of the Institute of Electrical Engineers, London, England.